

Presenting parabolic subgroups

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Abstract

Consider a relatively hyperbolic group G . We prove that if G is finitely presented, so are its parabolic subgroups. Moreover, a presentation of the parabolic subgroups can be found algorithmically from a presentation of G , a solution of its word problem, and generating sets of the parabolic subgroups. We also give an algorithm that finds parabolic subgroups in a given recursively enumerable class of groups.

Consider a relatively hyperbolic group G with parabolic subgroups H_1, \dots, H_n . It is well known that if each H_i is finitely generated (or finitely presented), then so is G . Osin showed conversely that if G is finitely generated, then so are H_1, \dots, H_n [Osi06, Prop. 2.27]. Whether finite presentation of G implies finite presentation of H_1, \dots, H_n is an important question raised by Osin in [Osi06, Problem 5.1].

On the algorithmic side, given a finite presentation of a relatively hyperbolic group G and a generating set of the parabolic subgroups, can one find a presentation of the parabolic subgroups?

We give a positive answer to these two questions.

Theorem 1. *Let G be a finitely presented group. Assume that G is hyperbolic relative to H_1, \dots, H_n . Then each H_i is finitely presented.*

Theorem 2. *There exists an algorithm that takes as input a finite presentation of a group G , a solution to its word problem, and a collection of finite subsets $S_1, \dots, S_n \subset G$, and that terminates if and only if G is hyperbolic relative to $\langle S_1 \rangle, \dots, \langle S_n \rangle$.*

In this case, the algorithm outputs a linear isoperimetry constant K for the corresponding relative presentation, a finite presentation for each of the parabolic subgroups $\langle S_i \rangle$, and says whether G is properly relative hyperbolic relative to $\langle S_1 \rangle, \dots, \langle S_n \rangle$ (i. e. $\langle S_i \rangle \subsetneq G$ for all i).

In this statement, the linear isoperimetry constant K is for the relative presentation X_∞ as defined in Section 1.2.

If one is not given generating sets of the parabolic subgroups, one can search for them, and require that they lie in some recursively enumerable class of groups.

Theorem 3. *There exists an algorithm as follows. It takes as input a finite presentation of a group G , a solution for its word problem, and a recursive*

class of finitely presented groups \mathcal{C} (given by a Turing machine enumerating presentations of these groups).

It terminates if and only if G is properly hyperbolic relative to subgroups that are in the class \mathcal{C} .

In this case, the algorithm outputs an isoperimetry constant K , a generating set and a finite presentation for each of the parabolic subgroups.

The Turing machine enumerating \mathcal{C} is a machine that enumerates some finite presentations, each of which represents a group in \mathcal{C} , and such that every group in \mathcal{C} has at least one presentation that is enumerated.

This paper can be seen as a continuation, extension, and precision, on the form and the substance of [Dah08]. It is based on the analysis of some Van Kampen diagrams in different *truncated* relative presentations. The main tool is Proposition 2.9 which says that if some relative presentation does not satisfy a linear isoperimetric inequality, then this shows up on some diagram of small area and small complexity.

Section 1 recalls definitions about isometric inequalities, introduces truncated relative presentations, and defines the complexity of a diagram. Section 2 contains the main technical results. Section 3 is devoted to corollaries. Theorems 1, 2, and 3 follow from Corollaries 3.3, 3.5 and 3.6.

1 Context

1.1 Linear isoperimetric inequalities

Consider a finitely generated group G , with an arbitrary (non necessarily finite) generating set S . A *presentation* of G over S is a set $\mathcal{R} \subset \mathbb{F}_S$ that normally generates the kernel of the natural map from the free group \mathbb{F}_S to G . The elements of \mathcal{R} are called *defining relations*, and we usually write $G = \langle S | \mathcal{R} \rangle$.

We say that a presentation is *triangular* if every defining relation has length 2 or 3 as word over the alphabet S^\pm . If one allows to increase the generating set, it is not restrictive to consider triangular presentations: from an arbitrary finite presentation, one can construct effectively a triangular one.

Consider $w \in \mathbb{F}_S$, viewed as a reduced word over the alphabet S^\pm . If w represents the trivial element of G (we write $w \stackrel{G}{=} 1$), the *area* of w for the presentation $G = \langle S | \mathcal{R} \rangle$, denoted by $\text{Area}(w)$, is the minimal number n such that w is the product in \mathbb{F}_S of n conjugates of elements of \mathcal{R} .

Given a word w such that $w \stackrel{G}{=} 1$, a *Van Kampen diagram* for w over the presentation $G = \langle S | \mathcal{R} \rangle$, is a simply connected planar 2-complex such that oriented edges are labeled by elements of S^\pm , such that reversing the orientation changes the label to its inverse, and such that every 2-cell has its boundary labeled by a cyclically reduced word conjugate to an element of $\mathcal{R} \cup \mathcal{R}^{-1}$, and such that the boundary of the diagram itself is labeled by w . Sometimes, we just say *cell* instead of 2-cell. It is well known that $\text{Area}(w)$ is the minimal number of 2-cells of Van Kampen diagrams for w . See [LS01, Section 5.1] for more details.

An *isoperimetric function* of a presentation $\langle S|\mathcal{R} \rangle$ is a function $f : \mathbb{N} \rightarrow \mathbb{N}$ such that for all $w \in \mathbb{F}_S$, $\text{Area}(w) \leq f(\text{length}(w))$. Note that if S is infinite, there are infinitely many words of a given length, and it may happen that no such function (with finite values) exists.

Our approach is based on the fact that a group is relatively hyperbolic if and only if it has a presentation of a particular kind with a linear isoperimetric function [Osi06], see Theorem 1.2 below. Another important fact is that the failure of a specific linear isoperimetric inequality can be observed in a set of words of controlled area (Gromov [Gro87], Bowditch [Bow95], Papasoglu [Pap95]).

Theorem 1.1 ([Pap95]). *Let $G = \langle S|\mathcal{R} \rangle$ be an arbitrary (non necessarily finite) triangular presentation of an arbitrary group.*

Assume that there is a word w over the alphabet S^\pm such that $w \stackrel{G}{=} 1$ and $\text{Area}(w) > K\text{length}(w)$. Then there exists a word w' over the alphabet S^\pm such that $w' \stackrel{G}{=} 1$, and such that

- $\text{Area}(w') \in [\frac{K}{2}, 240K]$
- $\text{Area}(w') > \frac{1}{2 \times 10^4} \text{length}(w')^2$.

1.2 Truncated and exact relative presentations

Since finite generation of a relatively hyperbolic group implies finite generation of its maximal parabolic subgroups [Osi06, Prop. 2.27], we always assume that relatively hyperbolic groups and their maximal parabolic subgroups are finitely generated.

Let G be a finitely presented group, and H_1, \dots, H_n be finitely generated subgroups of G . For each i , let S_i be a finite symmetric generating set of H_i . Consider a finite triangular presentation $G = \langle S|\mathcal{R} \rangle$ where S is a finite symmetric generating set of G containing each S_i , and \mathcal{R} is a finite set of triangular relations over S .

To introduce *truncated* relative presentations, we need auxiliary groups $\tilde{H}_1, \dots, \tilde{H}_n$, with generating sets $\tilde{S}_1, \dots, \tilde{S}_n$, and with epimorphisms $p_i : \tilde{H}_i \rightarrow H_i$ that map \tilde{S}_i bijectively to S_i . Informally, \tilde{H}_i is a group obtained from a presentation of H_i over S_i by removing some relations. *Exact* relative presentations will correspond to the case where each p_i is an isomorphism.

Let $\mathcal{T}(\tilde{H}_i) \subset \tilde{H}_i^*$ be the multiplication table of \tilde{H}_i , i. e. the set tuples of at most 3 elements of \mathbb{F}_{S_i} whose product is trivial in \tilde{H}_i . Thus, we have $(a, b, c) \in \mathcal{T}(\tilde{H}_i)$ if and only if $abc = 1$ in \tilde{H}_i .

Let $\hat{S} = S \sqcup \tilde{H}_1 \sqcup \dots \sqcup \tilde{H}_n$. To each element of \hat{S} corresponds naturally an element of G via the inclusion $S \subset G$ or via p_i . These elements of G form a generating set, in general infinite.

Given the initial presentation $G = \langle S|\mathcal{R} \rangle$, H_i and its generating set S_i , the auxiliary groups \tilde{H}_i and the epimorphisms $p_i : \tilde{H}_i \rightarrow H_i$, we associate the *truncated relative presentation* of G as follows:

$$G = \left\langle \hat{S} \mid \mathcal{R}', (\mathcal{T}(\tilde{H}_i))_{i=1\dots n} \right\rangle \quad (1)$$

where \mathcal{R}' consists of \mathcal{R} together with all two-letter relators of the form $\tilde{s}^{-1}p_i(\tilde{s})$ for $\tilde{s} \in \tilde{S}_i$, $(p_i(\tilde{s}))$ being an element of S). Obviously, this infinite presentation is indeed a triangular presentation of G .

We say that this presentation is truncated because only the multiplication table of \tilde{H}_i is included, and not the one of H_i (although all relations of H_i are consequences of \mathcal{R}'). We say that a truncated relative presentation as above is *exact* if for all i , $p_i : \tilde{H}_i \rightarrow H_i$ is an isomorphism.

We will be particularly interested in the following one-parameter family of truncated relative presentations X_ρ . Given G, H_i, S_i as above, and $\rho \in \mathbb{N} \cup \{\infty\}$, we define $\mathcal{R}_\rho(S_i)$ be the set of all words of length $\leq \rho$ on S_i that are trivial in H_i , $\tilde{H}_i = \langle S_i | \mathcal{R}_\rho(S_i) \rangle$, and $p_i : \tilde{H}_i \rightarrow H_i$ the obvious epimorphism. We define X_ρ the truncated relative presentation (1) corresponding to this data. In particular, X_∞ is an exact relative presentation, and if all H_i are finitely presented, then X_ρ and X_∞ coincide (as presentations) for ρ large enough.

Theorem 1.2 ([Osi06, Th. 1.7, Def. 2.29]). *G is hyperbolic relative to H_1, \dots, H_n if and only if the exact presentation X_∞ satisfies a linear isoperimetric inequality.*

The subgroups H_1, \dots, H_n of G are called the *maximal parabolic subgroups*. Since there is no risk of confusion, we will simply call them *parabolic subgroups*.

Remark 1.3. Osin includes all words of any length in the multiplication table. One easily checks that this does not change the result.

In section 3, we are going to prove that if X_∞ satisfies a linear isoperimetric inequality, so does X_ρ for ρ large enough. This will easily imply that parabolic subgroups are finitely presented.

1.3 Complexities

Since X_ρ is an infinite presentation, it is convenient to have a measure of complexity for letters and words on \hat{S} . Recall that $\hat{S} = S \sqcup \tilde{H}_1 \sqcup \dots \sqcup \tilde{H}_n$. For $a \in \tilde{H}_i$, we denote by $|\tilde{a}|_{\tilde{S}_i}$ the word length of a relative to the generating set \tilde{S}_i . We define the *complexity* $\|a\|$ of $a \in \hat{S}$ as 1 if $a \in S$, and as $|\tilde{a}|_{\tilde{S}_i}$ if $a \in \tilde{H}_i$.

Given a word $w = a_1 \cdots a_n$ over \hat{S} , we define

- $\text{length}(w) = n$
- $\|w\|_1 = \sum_{i=1}^n \|a_i\|$
- $\|w\|_\infty = \max_{i=1}^n \|a_i\|$

Note that if w is a one-letter word, then $\|w\|_1 = \|w\|_\infty = \|w\|$.

Similarly, if D is a diagram (or a path) whose edges are labeled by elements of \hat{S} , we define $\|D\|_1$ and $\|D\|_\infty$ as the sum and the maximum of the complexities of the labels of its edges. For a labeled path p , $\text{length}(p)$ denotes its number of edges, and $\text{Area}(D)$ denotes the number of 2-cells of a diagram D .

2 Diagrams

The goal of this section is to prove that if X_ρ does not satisfy a linear isoperimetric inequality, this shows up on diagrams of small area and small complexity (Proposition 2.9).

2.1 Vocabulary

Thickness. Let D be a Van Kampen diagram over the presentation X_ρ (ρ being fixed in $\mathbb{N} \cup \{\infty\}$). We denote by $D_{\text{thick}} \subset D$ the union of all 2-cells, and of all vertices and edges that are contained in the boundary of a 2-cell. We say that D is *thick* if $D = D_{\text{thick}}$ i. e. if every edge lies in the boundary of a 2-cell.

Clusters. We define cells of type \mathcal{R}' (resp. of type \tilde{H}_i) as those labeled by a word of \mathcal{R}' (resp. by a word in $\mathcal{T}_{S_i}(\tilde{H}_i)$). Note that two cells of type \tilde{H}_i and \tilde{H}_j cannot share an edge if $i \neq j$.

Two cells of the same type \tilde{H}_i and sharing an edge are said *cluster-adjacent*. A *cluster* is an equivalence class for the transitive closure of this relation. All 2-cells of a cluster have the same type \tilde{H}_i , which we define as the type of the cluster. We identify a cluster with the closure C of the 2-cells it is made of. Note that clusters are contained in D_{thick} .

If C is a cluster, we denote by ∂C (its boundary) the union of closed edges of C that are in only one 2-cell of C .

Remark 2.1. Note that for any cluster C , any edge in $\partial C \setminus \partial D$ has complexity 1. Indeed, the 2-cell of $D \setminus C$ containing this edge is labeled by a relator $\tilde{s}^{-1}p_i(\tilde{s})$ for some $\tilde{s} \in \tilde{S}_i$.

2.2 Simply connected clusters, standard filling

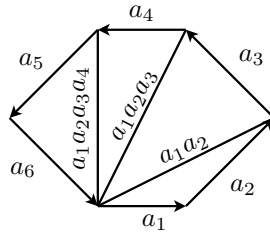


Figure 1: Standard filling.

Note that a cluster C (as a subset of the plane) is simply connected if and only if C is a disk and ∂C is an embedded circle in the plane. We will mostly deal with diagrams whose clusters are simply connected.

Consider a simply connected cluster C , with ∂C labeled by the cyclic word a_1, \dots, a_n (where each $a_j \in \tilde{H}_i$). A *standard filling* of ∂C is a diagram with

boundary ∂C , with $n - 2$ triangles as in figure 1, all whose vertices are in ∂C , and whose interior edges are labeled by $a_1 \dots a_j$ for $j \leq n - 2$, where $a_1 \dots a_j$ is viewed as an element of \tilde{H}_i .

Lemma 2.2. *If C is an arbitrary simply connected cluster, then $\|\partial C\|_1 \leq 3\text{Area}(D) + \|\partial D\|_1$,*

If C is standardly filled, then $\text{Area}(C) = \text{length}(\partial C) - 2$, and $\|C\|_\infty \leq \|\partial C\|_1$.

Proof. Let us partition ∂C into edges that are in ∂D and inner edges. There are at most $\text{length}(\partial C) \leq 3\text{Area}(D)$ inner edges, each of which is of complexity 1, by Remark 2.1. The sum of complexities of the edges in ∂D is bounded by $\|\partial D\|_1$. This proves the first assertion. The second assertion is clear from the definition. \square

Remark 2.3. If C is any cluster, then $\text{Area}(C) \geq \text{length}(\partial C) - 2$. Indeed, Denoting by F , E_{int} , E_{ext} the number of 2-cells, interior edges and boundary edges, connectedness of the dual graph implies $F - 1 \leq E_{\text{int}}$. Since cells of C have at most 3 sides, $2E_{\text{int}} + E_{\text{ext}} \leq 3F$. It follows that $E_{\text{ext}} \leq F + 2$ as required.

The following lemma shows that in many situations, clusters are simply connected.

Lemma 2.4. *Let w be a word over \hat{S} defining the trivial element in G . Let D be a minimal Van Kampen diagram for w over the presentation X_ρ . Assume that $\rho \geq 3\text{Area}(D)$.*

If D is chosen among diagrams for w over X_ρ to minimize successively the area, and the number of 2-cells of type \mathcal{R}' , then every cluster of D is simply connected.

Assume either that D is as above and that all its clusters are standardly filled, or that D minimizes successively the area, the number of 2-cells of type \mathcal{R}' and $\|D\|_\infty$. Then

$$\|D\|_\infty \leq 3\text{Area}(D) + \|w\|_1.$$

Proof. Assume by contradiction that there exists a cluster C of type \tilde{H}_i that is not simply connected. Then there is a simply connected subdiagram $D' \subset D$ such that edges of $\partial D'$ are all in $\partial C \setminus \partial D$. Since edges of $\partial D'$ lie in a 2-cell, $\text{length}(\partial D') \leq 3\text{Area}(D)$. Moreover $\|\partial D'\|_\infty = 1$, since by Remark 2.1, every edge in $\partial C \setminus \partial D$ has complexity 1. Thus, $\|\partial D'\|_1 \leq 3\text{Area}(D)$. Since $\rho \geq 3\text{Area}(D)$, the definition of X_ρ says that the word labeled by $\partial D'$ is trivial in \tilde{H}_i . One can then replace the subdiagram bounded by c by a diagram with same combinatorics, and with cells of type \tilde{H}_i . This contradicts the minimality of D for the number of 2-cells of type \mathcal{R}' . It follows that all clusters of D are simply connected.

Assume now that all clusters are standardly filled. By Lemma 2.2, for each cluster C , $\|C\|_\infty \leq \|\partial C\|_1 \leq 3\text{Area}(D) + \|w\|_1$. Since each edge of D_{thick} of

complexity at least 2 is contained in a cluster, this implies that $\|D_{\text{thick}}\|_{\infty} \leq 3\text{Area}(D) + \|w\|_1$.

Finally, assume that D minimizes successively the area, the number of 2-cells of type \mathcal{R}' and $\|D\|_{\infty}$. Since clusters of D are simply connected, we can modify D to a diagram D' whose clusters are standardly filled, and having the same area and the same number 2-cells of type \mathcal{R}' as D . In particular, $\|D\|_{\infty} \leq \|D'\|_{\infty}$. By the argument above, $\|D'\|_{\infty} \leq 3\text{Area}(D) + \|w\|_1$ which concludes the proof. \square

2.3 Complicated clusters

A cluster C is said to be *complicated* if $\partial C \cap \partial D$ contains at least two edges.

Lemma 2.5. *Assume that D is a Van Kampen diagram, and $C \subset D$ is a simply connected cluster.*

If C is not complicated, then $\|\partial C\|_{\infty} \leq \text{length}(\partial C)$, $\|\partial C\|_1 \leq 2\text{length}(\partial C)$.

Proof. Denote by \tilde{H}_i the type of the cluster C , so that edges of C are labeled by elements of \tilde{H}_i . If C is not complicated, all edges of ∂C but one have complexity 1. The cluster being simply connected, the label of the remaining edge has the same image in \tilde{H}_i as a product of $\text{length}(\partial C) - 1$ elements of S_i . Therefore, this edge has complexity at most $\text{length}(\partial C) - 1$. It follows that $\|\partial C\|_{\infty} \leq \text{length}(\partial C)$, and $\|\partial C\|_1 \leq (\text{length}(\partial C) - 1) + \sum_{e \in \partial C} 1$. This proves the lemma. \square

Lemma 2.6 (See also [Osi06, Lemma 2.27]). *Let D be a Van Kampen diagram whose clusters are simply connected, non complicated, and standardly filled.*

Then $\|D_{\text{thick}}\|_{\infty} \leq 6\text{Area}(D)$.

Proof. Any edge of D_{thick} is either contained in a cell of type \mathcal{R}' (it has complexity 1) or in a cluster C . Since the number of edges of D that lie in the boundary of a 2-cell is bounded by $3 \times \text{Area}(D)$, we have $\text{length}(\partial C) \leq 3 \times \text{Area}(D)$. Since C is not complicated, $\|C\|_{\infty} \leq 6 \times \text{Area}(D)$ by Lemma 2.5. The lemma follows. \square

2.4 Arcs-of-clusters and pieces

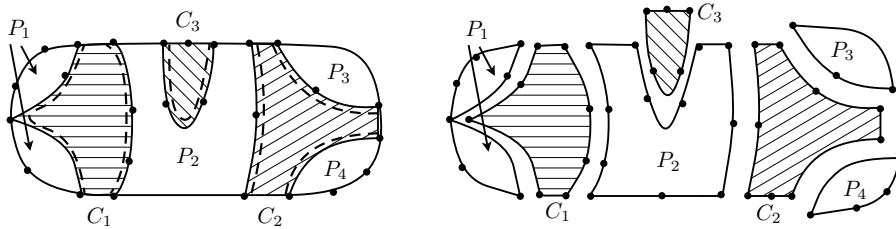


Figure 2: 3 complicated clusters, 4 regular pieces, and 6 arcs-of-clusters

Consider a diagram D whose clusters are simply connected. An *arc-of-cluster* is a maximal subpath $c \subset \partial C$ for some complicated cluster C that does not contain any edge of ∂D (see Figure 2). Since ∂C is an embedded circle, each arc-of-circle c is an embedded arc with endpoints in ∂D , and $c \cap \partial D$ contains no edge, but it may contain vertices distinct from its endpoints.

We define *regular pieces* of D as the connected components of $D \setminus \mathring{C}$ where \mathring{C} denotes the interior in D of the union of all complicated clusters in D (edges in $\partial D \cap \partial C$ for some complicated cluster are in \mathring{C}), see Figure 2. Regular pieces and complicated clusters are called *pieces*.

Here is an alternative definition. For each complicated cluster C , consider properly embedded arcs with endpoints in ∂D , that are very close and parallel to each arc-of-cluster, obtained by pushing inside C the arcs-of-clusters. Let \mathcal{A} be the union of such embedded arcs when C ranges over all complicated clusters. Then connected components of $S \setminus \mathcal{A}$ are in one-to-one correspondence with pieces. On figure 2, \mathcal{A} is represented by dotted lines.

Clearly, the set of pieces induces a partition of the set of 2-cells of D . There is a natural *incidence graph* \mathcal{G} for this partition, whose vertices are the pieces, whose edges are the arcs-of-clusters, the two endpoints of an edge being the cluster and the regular piece on both sides of the corresponding arc-of-cluster.

Lemma 2.7. *Let D be a Van Kampen diagram, and assume that any cluster of D is simply connected.*

The incidence graph \mathcal{G} is a bipartite tree and the degree of a vertex v associated to a complicated cluster C is at most the number of edges in $\partial D \cap \partial C$, with strict inequality when the vertex is v is a leaf of the tree \mathcal{G} .

Proof. The graph is bipartite by definition. It is connected because D is. Since every arc-of-cluster separates D , every edge of the incidence graph disconnects it. This proves that \mathcal{G} is a tree.

Consider a vertex v associated to a complicated cluster C . The degree of v is, by definition, the number of arcs-of-clusters on ∂C . Since C is simply connected, ∂C is an embedded circle, and since C is complicated, ∂C contains an edge of ∂D . By maximality in the definition of arc-of-clusters, each such arc is followed in ∂C (with a chosen fixed orientation) by an edge of $\partial C \cap \partial D$. This association, which is clearly one-to-one, ensures the bound on the degree.

Finally, if v is a leaf of \mathcal{G} , its degree is 1 and $\partial D \cap \partial C$ contains at least 2 edges because C is complicated. \square

The following result of [Dah08] was, to some extent, left to the reader. We include a proof.

Lemma 2.8. *Let D be a Van Kampen diagram. If every cluster is simply connected, then the number of pieces, and the number of arc-of-clusters are both bounded by $\text{length}(\partial D)$.*

Proof. The number N of pieces is the number of vertices of the incidence graph \mathcal{G} . Since \mathcal{G} is a tree, $N = E + 1$ where E is the number of edges of \mathcal{G} , i. e. the number of arcs-of-clusters. Denote by v_C the vertex corresponding to a

cluster C , by $d(v_C)$ its degree, and by V_{cl} the set of all vertices of \mathcal{G} corresponding to clusters. Since \mathcal{G} is bipartite, $E = \sum_{v_C \in V_{cl}} d(v_C)$. By lemma 2.7, $d(v_C)$ is bounded by the number $e(C)$ of edges of $\partial C \cap \partial D$. Therefore $E \leq \sum_{v_C \in V_{cl}} e(C) \leq \text{length}(\partial D)$.

Finally, if some v_C is a leaf of \mathcal{G} , this last inequality is a strict inequality, which yields $N = E + 1 \leq \text{length}(\partial D)$. There remains the case where some leaf of \mathcal{G} is a regular piece B . This means that $\partial B = \alpha \cup \beta$ where α is an arc-of-cluster, and β is a path in ∂D . Since clusters are simply connected, the endpoints of α are distinct, so β contains at least an edge. This implies that $\sum_{v_C \in V_{cl}} e(C) < \text{length}(\partial D)$, and concludes the lemma. \square

2.5 Reduction to diagrams of small complexity

We are now ready to state and prove the main statement of this section. It claims that if X_ρ does not satisfy a linear isoperimetric inequality, this shows up on diagrams of small area (this is Papasoglu's theorem) and small complexity.

Proposition 2.9 ([Dah08, Prop. 1.5]). *Let $K \geq 10^6$ and $\rho \in \mathbb{N} \cup \{\infty\}$, $\rho \geq 3 \times 240K$.*

Assume that X_ρ fails to satisfy a linear isoperimetric inequality of constant K (that is, there exists a word w over the alphabet \hat{S} such that $\text{Area}(w) > K \text{length}(w)$).

Then, there exists a word w'' over the alphabet \hat{S} , and a minimal Van Kampen diagram D'' (over X_ρ) for w'' , such that

- (1) $\text{Area}(D'') \leq 240K$
- (2) $\|D''\|_\infty \leq 2.10^6 K^2$
- (3) $\text{Area}(D'') > \frac{\sqrt{K}}{600} \text{length}(\partial D'')$.

Proof. The first step is to apply Papasoglu's Theorem 1.1 to the presentation X_ρ to obtain a word w' over \hat{S} for which $K/2 \leq \text{Area}(w') \leq 240K$, and $\text{Area}(w') > \frac{1}{2 \times 10^4} \text{length}(w')^2$.

Using $\sqrt{\text{Area}(w')} > \frac{\text{length}(\partial w')}{\sqrt{2 \times 10^4}}$ and $\text{Area}(w') \geq K/2$, we get

$$\text{Area}(w') > \sqrt{\frac{\text{Area}(w')}{2 \times 10^4}} \text{length}(w') \geq \frac{\sqrt{K}}{200} \times \text{length}(w').$$

Choose a diagram D' among minimal area diagrams over X_ρ for w' so that the number of 2-cells of type \mathcal{R}' is minimal. We claim that up to changing w' , we can assume that D' is thick i. e. all edges lie in the boundary of a 2-cell. Indeed, if all connected components A'_1, \dots, A'_l of D'_{thick} satisfy $\text{Area}(A'_i) \leq \frac{\sqrt{K}}{200} \times \text{length}(\partial A'_i)$, then

$$\text{Area}(D') = \sum_i \text{Area}(A'_i) \leq \frac{\sqrt{K}}{200} \sum_i \text{length}(\partial A'_i) \leq \frac{\sqrt{K}}{200} \times \text{length}(w')$$

a contradiction. It follows that some component A'_i satisfies $\text{Area}(A'_i) > \frac{\sqrt{K}}{200} \times \text{length}(\partial A'_i)$. Obviously, $\text{Area}(A'_i) \leq \text{Area}(D') \leq 240K$, and A'_i is a diagram for $\partial A'_i$ that minimizes the area and the number of cells of type \mathcal{R}' (if not, substituting a diagram of smaller area for $\partial A'_i$ in D' contradicts minimality of D'). This proves that we can assume that D' is thick.

We do not have any control on the complexity of a diagram filling w' yet. By choice of ρ , Lemma 2.4 shows that the clusters of D' are simply connected. We can modify D' and assume that all clusters are standardly filled. By Remark 2.3, D' still minimizes area and the number of cells of type \mathcal{R}' . By Lemma 2.8, the number of pieces in the decomposition into complicated clusters and regular pieces is at most $\text{length}(\partial D')$.

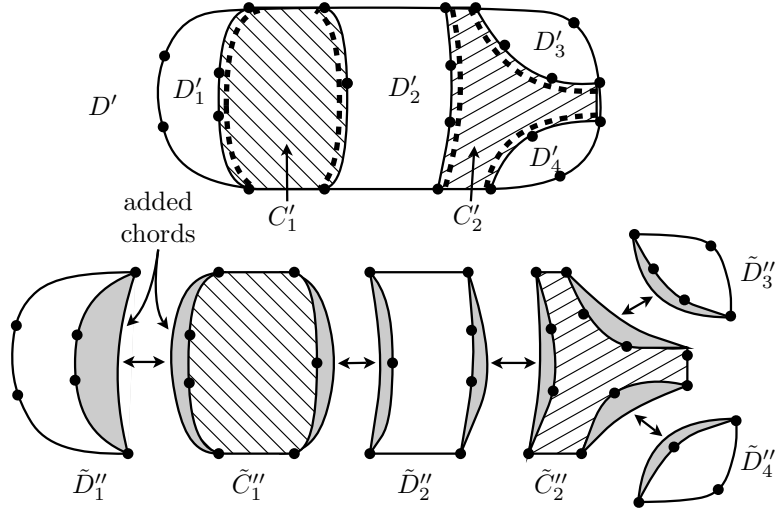


Figure 3: Adding chords to the pieces of D' , and regluing them together

Let C'_1, \dots, C'_s be the complicated clusters of D' , and D'_1, \dots, D'_r be the regular pieces. We construct new diagrams C''_i , D''_j , and \tilde{C}''_i , \tilde{D}''_j from C'_i , D'_j by first *adding chords*, then by changing the triangulation as follows (see Figure 3).

Fix a complicated cluster C'_k of D' , and denote by \tilde{H}_i its type. Its boundary $\partial C'_k$ is a union of pairwise disjoint arcs-of-clusters, together with arcs in $\partial D'$. Consider an arc-of-cluster $c \subset \partial C'_k$ whose edges are labeled by elements a_1, \dots, a_n of \tilde{H}_i , and let $a_c = a_1 \dots a_n \in \tilde{H}_i$ be their product. We glue along c a standardly filled disk with boundary labeled by $a_1, \dots, a_n, a_c^{-1}$. We name the new edge labeled by a_c^{-1} a *chord*. Performing this operation for each arc-of-cluster, we get a disk C''_k made of cells of type \tilde{H}_i . Finally, we change the triangulation of this disk to a standard filling, and we call \tilde{C}''_k the obtained diagram. Note that $\text{Area}(\tilde{C}''_k) \leq \text{length}(\partial \tilde{C}''_k) - 2$.

Now, we perform a similar operation for each regular piece D'_j . For each arc-of-cluster $c \subset \partial D'_j$ labeled by $a_1, \dots, a_n \in \tilde{H}_i$ (now, the type \tilde{H}_i may depend on

c), we define $a_c = a_1 \dots a_n \in \tilde{H}_i$, and glue to C'_k along c a new cluster of type \tilde{H}_i , standardly filled, whose boundary is labeled by $a_1, \dots, a_n, a_c^{-1}$. Since the filling is standard, the area of the added cluster is $(n+1) - 2 = \text{length}(c) - 1$. Performing this operation for each arc-of-cluster, we get the new diagram D''_j . Finally, we take for \tilde{D}''_j a diagram with boundary $\partial D''_j$, and minimizing successively the area and the number of 2-cells of type \mathcal{R}' .

We are going to bound $\|\tilde{D}''_j\|_\infty$ by first bounding $\|D''_j\|_\infty$. Since all complicated clusters of D' are C'_1, \dots, C'_s , D''_j has no complicated cluster coming from D' . The newly created clusters in D''_j have just one edge in $\partial D''_j$, so are not complicated. Therefore, clusters of D''_j are not complicated, simply connected, and standardly filled. Since D' is thick, so is D''_j . Applying Lemma 2.6 to D''_j , we get $\|D''_j\|_\infty \leq 6 \times \text{Area}(D''_j) \leq 6 \times 240K$.

In particular, $\|\partial \tilde{D}''_j\|_\infty = \|\partial D''_j\|_\infty \leq 6 \times 240K$, and since D''_j is thick, $\|\partial D''_j\|_1 \leq 3 \text{Area}(D''_j) \|\partial D''_j\|_\infty \leq 18 \times (240K)^2$. Applying Lemma 2.4 to \tilde{D}''_j , we get

$$\|\tilde{D}''_j\|_\infty \leq 3 \text{Area}(D''_j) + \|\partial D''_j\|_1 \leq 3 \times 240K + 18 \times (240K)^2 \leq 2.10^6 K^2.$$

This proves that for all $j \in \{1, \dots, r\}$, \tilde{D}''_j satisfies assertions (1) and (2) of the proposition.

We now prove that one of the diagrams \tilde{D}''_j , $j = 1, \dots, r$ must satisfy (3). Assume by contradiction that for all $j \in \{1, \dots, r\}$, $\text{Area}(\tilde{D}''_j) \leq \frac{\sqrt{K}}{600} \text{length}(\partial \tilde{D}''_j)$. Note that \tilde{C}''_k satisfies this inequality as well. Indeed, $\text{Area}(\tilde{C}''_k) \leq \text{length}(\partial \tilde{C}''_k)$, and by assumption, $K \geq 10^6$ so $\frac{\sqrt{K}}{600} \geq 1$.

Gluing together the diagrams $\tilde{D}''_1, \dots, \tilde{D}''_r$ and $\tilde{C}''_1, \dots, \tilde{C}''_s$ pairwise along the two chords corresponding to a given arc-of-cluster as shown on Figure 3, we get another (non necessarily minimal) Van Kampen diagram \tilde{D}' for w' .

We have

$$\begin{aligned} \text{Area}(D') &\leq \text{Area}(\tilde{D}') = \sum_j \text{Area}(\tilde{D}''_j) + \sum_k \text{Area}(\tilde{C}''_k) \\ &\leq \frac{\sqrt{K}}{600} \left(\sum_j \text{length}(\partial \tilde{D}''_j) + \sum_k \text{length}(\partial \tilde{C}''_k) \right) \\ &\leq \frac{\sqrt{K}}{600} \left(\text{length}(\partial D') + 2n_a \right) \end{aligned}$$

where n_a is the number of arcs-of-clusters in D' . By lemma 2.8, $n_a \leq \text{length}(\partial D')$, so $\text{Area}(D') \leq \frac{\sqrt{K}}{200} \times \text{length}(\partial D')$, thus contradicting the property of D' established at the beginning of the proof. \square

3 Consequences

Corollary 3.1. *Assume that X_∞ satisfies a linear isoperimetric inequality of factor $K \geq 10^6$. Let $K' = (600K)^2$ and $\rho(K) = 10^{26} K^5$. Then for all $\rho \geq \rho(K)$,*

X_ρ satisfies a linear isoperimetric inequality of factor K' .

Before proving the corollary, we need to relate more explicitly the presentations X_ρ and X_∞ . Consider $\hat{S}_\rho = S \sqcup \tilde{H}_1 \sqcup \dots \sqcup \tilde{H}_n$ and $\hat{S}_\infty = S \sqcup H_1 \sqcup \dots \sqcup H_n$ the corresponding generating sets. The morphisms $p_i : \tilde{H}_i \rightarrow H_i$ induce an obvious map $p : \hat{S}_\rho \rightarrow \hat{S}_\infty$ that is the identity on S and maps \tilde{H}_i to H_i through p_i . If $w = a_1 \dots a_n$ is a word over \hat{S}_ρ , we denote by $p(w) = p(a_1) \dots p(a_n)$ the corresponding word over \hat{S}_∞ . Clearly, if w is any relator of X_ρ , $p(w)$ is a relator of X_∞ . It follows that given any diagram D over X_ρ for a word w , one gets a new diagram $p_*(D)$ for $p(w)$ over X_∞ by applying the map p to all the labels of all edges of D .

On the other hand, p_i induces a bijection between the balls of radius $\rho/2$ of \tilde{H}_i and H_i , whose inverse we denote by p_i^{-1} . Similarly, we denote by p^{-1} the inverse of the restriction of $p : \hat{S}_\rho \rightarrow \hat{S}_\infty$ to the set of elements of complexity at most $\rho/2$. Now, if $a, b, c \in H_i$ are in the ball of radius $\rho/3$ of H_i and satisfy $abc = 1$ in H_i , then $p_i^{-1}(a)p_i^{-1}(b)p_i^{-1}(c) = 1$ in \tilde{H}_i . This means that if some diagram D over X_∞ for w satisfies $\|D\|_\infty \leq \rho/3$, then the diagram $p_*^{-1}(D)$ (with obvious notations) is a diagram over X_ρ for $p^{-1}(w)$.

Proof of Corollary 3.1. Assume that X_ρ fails to satisfy the predicted isoperimetric inequality (of factor K'), and argue towards a contradiction. By Proposition 2.9, there is a word w'' representing the trivial element, with a diagram D'' , minimal over the presentation X_ρ , of area at most $240K'$, and complexity $\|D''\|_\infty \leq 2.10^6 K'^2$ and such that $\text{Area}(D'') > K \times \text{length}(w'')$.

Consider the map $p : \hat{S}_\rho \rightarrow \hat{S}_\infty$ described above. Choose D''_0 among diagrams for $p(w'')$ in the presentation X_∞ , in order to minimize successively the area, the number of 2-cells of type \mathcal{R}' , and the complexity $\|D''_0\|_\infty$. Since X_∞ satisfies a linear isoperimetric inequality of factor K , $\text{Area}(D''_0) < \text{Area}(D'') \leq 240K'$. By Lemma 2.4, $\|D''_0\|_\infty \leq 720K' + \|p(w'')\|_1$. On the other hand,

$$\begin{aligned} \|p(w'')\|_1 &\leq \|w''\|_1 \leq \text{length}(w'') \|D''\|_\infty \leq \frac{1}{K} \text{Area}(D'') \times 2.10^6 K'^2 \\ &\leq \frac{240K' \times 2.10^6 K'^2}{K} \leq 3.10^{25} K^5. \end{aligned}$$

Since $K \geq 10^6$, $720K' \leq 10^9 K^2 \leq K^5$. By hypothesis on ρ , we see that $\|D''_0\|_\infty \leq \rho/3$. It follows that $p_*^{-1}(D)$ is a diagram over X_ρ for w'' , of area $< \text{Area}(D'')$, a contradiction. \square

Lemma 3.2. *Assume that X_ρ satisfies a linear isoperimetric inequality of factor K' with $\rho \geq \max(3K', 2)$.*

Then $p_i : \tilde{H}_i \rightarrow H_i$ is an isomorphism. In particular, H_i is finitely presented, with a presentation whose defining relations are of length $\leq \rho$.

Proof. Assume by contradiction that $p_i : \tilde{H}_i \rightarrow H_i$ is not injective, and consider $a \in \ker p_i \setminus \{1\}$. Then a is a generator of the presentation X_ρ that represents the trivial element of G . Note that since $\rho > 1$, $a \notin \tilde{S}_i$. Therefore, there exists

a Van Kampen diagram D over X_ρ whose boundary consists of a single edge e labeled a , and whose area is at most K' . We choose a diagram for a over X_ρ in order to minimize successively the area, the number of 2-cells of type \mathcal{R}' , and $\|D\|_\infty$. Since $\rho \geq 3K'$, Lemma 2.4 implies that clusters of D are simply connected. Since $a \notin \tilde{S}_i$, e lies in a cluster C of type \tilde{H}_i . But since C is simply connected, and since a cluster of type \tilde{H}_i involves only relations of \tilde{H}_i , we get that a is trivial in \tilde{H}_i , a contradiction. \square

Corollary 3.3. *Assume that X_∞ satisfies a linear isoperimetric inequality of factor K .*

Then the subgroups P_i are finitely presented, with a presentation whose defining relations are of length $\leq \rho(\max(K, 10^6))$.

Proof. Without loss of generality, we can assume $K \geq 10^6$. By Corollary 3.1, $X_{\rho(K)}$ satisfies a linear isoperimetric inequality of factor $K' = (600K)^2$. Lemma 3.2 concludes. \square

Lemma 3.4 (see also [Osi06, Lemma 5.4]). *Assume that X_∞ satisfies a linear isoperimetric inequality of factor K .*

If $s \in S$ represents an element a of H_i , then $\|a\| \leq 12K$.

Proof. The word $w = sa$ is a word of length 2 over X_∞ . If it represents the trivial element in G , then there is a Van Kampen diagram D over X_ρ whose boundary is a path of length 2 labeled sa , and whose area is at most $2K$. We choose D among minimal area diagrams over X_∞ for w so that the number of 2-cells of type \mathcal{R}' is minimal. Since $\rho = \infty$, Lemma 2.4 implies that clusters of D are simply connected, and we can assume that they are standardly filled.

Note that there is no complicated cluster as only the edge labeled a of ∂D can be in a cluster. By Lemma 2.6, this implies that $\|D_{\text{thick}}\|_\infty \leq 12K$, so $\|a\| \leq 12K$. \square

We obtain the following improvement of [Dah08]:

Corollary 3.5. *There exists an algorithm that takes as input a finite presentation of a group G , a solution of its word problem, and a collection of finite subsets $S_1, \dots, S_n \subset G$, and that terminates if and only if G is hyperbolic relative to $\langle S_1 \rangle, \dots, \langle S_n \rangle$.*

In this case, produces an isoperimetry constant K for the presentation X_∞ , a finite presentation for each of the parabolic subgroups, and says whether G is parabolic (i. e. $G = \langle S_i \rangle$ for some i).

Proof. For a fixed $K \geq 10^6$, we consider all diagrams D over X_∞ such that $\|D\|_\infty \leq B = 2 \cdot 10^6 K^2$ and $\text{Area}(D) \leq 240K$. There are only finitely many. The word problem in G allows to list all relators of $\langle S_i \rangle$ of length at most $3B$, and hence to list these diagrams. Out of this list, we make the list $\mathcal{W}(K)$ of words labeling the boundaries of these diagrams.

We claim that given $w \in \mathcal{W}(K)$, we can compute $\text{Area}(w)$. Indeed, let D' be a diagram for w chosen to minimize area, the number of cells of type \mathcal{R}' ,

and $\|D'\|_\infty$. By Lemma 2.4, $\|D'\|_\infty \leq 3\text{Area}(D') + \|w\|_1 \leq 720K + \|w\|_1$. We can compute an upper bound $M \geq 720K + \|w\|_1$ for $\|D'\|_\infty$, and we can list all diagrams D' with $\text{Area}(D') \leq 240K$ and $\|D'\|_\infty \leq M$ whose boundary is w . We can then compute $\text{Area}(w)$ as the minimal area of these diagrams.

Now we can check whether $\text{Area}(w) \leq \frac{\sqrt{K}}{600}\text{length}(w)$ for all $w \in \mathcal{W}(K)$. If this is not the case, the algorithm increments K and starts over.

If this is the case, then by Proposition 2.9, X_∞ satisfies isoperimetric inequality of factor K , and the algorithm stops. It outputs K , and gives as set of relators for $\langle S_i \rangle$, the set of all words of length $\leq \rho(K)$ that are trivial in G ; this can be done using the word problem in G , and this is indeed a presentation of $\langle S_i \rangle$ by Lemma 3.3. To check whether $G = \langle S_i \rangle$, one needs to check whether each $s \in S$ represents an element $a \in \langle S_i \rangle$. Lemma 3.4 bounds the complexity of a , and we can try all possibilities for a using the word problem.

If X_∞ does satisfy a linear isoperimetric inequality of factor K_0 , then the process will obviously stop when K will reach a value greater than $(600K_0)^2$. \square

Corollary 3.6. *There exists an algorithm as follows. It takes as input a finite presentation of a group G , a solution for its word problem, and a recursive class of finitely presented groups \mathcal{C} (given by a Turing machine enumerating them). It terminates if and only if G is properly hyperbolic relative to subgroups that are in the class \mathcal{C} .*

In this case, the algorithm produces an isoperimetry constant K , a generating set and a finite presentation for each of the parabolic subgroups.

Proof. First, enumerate all possible presentations of groups in \mathcal{C} using the Turing machine given as input, and Tietze transformations. In parallel, list all possible families of finite subsets $\mathcal{S} = (S_1, \dots, S_n)$ of G . For each of them, run in parallel the algorithm of Corollary 3.5 that stops if G is hyperbolic relative to $\langle S_1 \rangle, \dots, \langle S_n \rangle$ and outputs a presentation of $\langle S_i \rangle$ in this case, and says whether G is parabolic. Get rid of those \mathcal{S} such that G is parabolic.

Then stop if at some point, one sees that in some of the produced presentations, $\langle S_i \rangle$ lie in \mathcal{C} . \square

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